Technical Memorandum 527249322-1

Structural mobility of a simply supported plate

Project: Near-field Acoustical Holography - a new sensor concept for methods of active noise reduction

Dr.-Ing. Steffen Ungnad

27.02.2024

Faculty of Mechanical and Civil Engineering

1 Introduction

This memorandum is the first in a series of documents that reflect the content of the work plan in [1]. The initial step is to derive the structural mobility of a thin plate. This content is well documented in the literature but will be required at various points during the course of the project and is therefore created here as a separate document. Beyond that, a simple analytical model is needed to develop an Active Structural Acoustic Control (ASAC) sensor principle using Nearfield Acoustical Holography (NAH), see [1]. An experimentally feasible approach is a thin vibrating plate in an infinite rigid baffle, where one of Rayleigh's integral equations is used to calculate the radiated sound pressure from a finite excitation within the baffle plane. This excitation function comes from the plate's normal structural vibration velocity, assuming no backward coupling between the sound pressure and the vibration velocity. Thus, it is assumed that the structural velocity matches the particle velocity on the interface between fluid and plate. In-vacuo structural modes with simply supported boundaries will be used, and this report will derive related equations for the structural mobility.

2 Structural mobility of a simply supported plate

The transversal structural mobility $Y_{mn}(x,y)$ of a simply supported plate with point excitation at position $[x', y']$, for the mode with modal indices m and n, is derived according to [2]†. Therefore, generalized coordinates q_{mn} and non-conservative forces Q_{mn} together with the resulting Lagrange's equation, i. e.,

$$\frac{d}{dt}\left(\frac{\partial T_{mn}}{\partial \dot{q}_{mn}}\right) - \frac{\partial T_{mn}}{\partial q_{mn}} + \frac{\partial V_{mn}}{\partial q_{mn}} = Q_{mn}, \tag{1}$$

are used. The time is denoted by t, with short-hand notation $(\dot{\ldots})$ for a time derivative. The modal kinetic and potential energies in generalized coordinates are given by

$$T_{mn} = \frac{\varrho h}{2}\frac{ab}{4}\dot{q}_{mn}^2, \tag{2}$$

and

$$V_{mn} = \frac{\omega_{mn}^2}{2}\frac{ab}{4}q_{mn}^2. \tag{3}$$

The angular eigenfrequency is calculated by

$$\omega_{mn} = \sqrt{\frac{D}{\varrho h}}(k_m^2 + k_n^2). \tag{4}$$

$k_m = m\pi/a$ and $k_n = n\pi/b$ represent modal wavenumbers according to the plate's length a and width b. The bending stiffness of the plate is defined as

† p. 20, p. 119, pp. 238-241, p.309

$$D = \frac{Eh^3}{12(1-\nu^2)}, \tag{5}$$

with density ϱ, thickness h, Young's modulus E, and Poisson's ratio ν. Inserting Eqs. (2) and (3) in Eq. (1) yields

$$\frac{d}{dt^2} q_{mn} + \omega_{mn}^2 q_{mn} = \frac{1}{\varrho h} \frac{4}{ab} Q_{mn}. \tag{6}$$

The transversal deflection w of the plate, fulfilling the simply supported boundary condition, is represented by the double series

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin(k_m x) \sin(k_n y) \,. \tag{7}$$

Accordingly, the generalized force at time t' takes the form

$$Q_{mn} = F \sin(k_m x') \sin(k_n y') \, e^{-\mathrm{i}\omega t'}. \tag{8}$$

The general solution of Eq. (6), by considering only the vibration due to the point force F, is

$$q_{mn} = \frac{F}{\varrho h \omega_{mn}} \frac{4}{ab} \sin(k_m x') \sin(k_n y') \int_0^t e^{-\mathrm{i}\omega t'} \sin\big(\omega_{mn}(t - t')\big) \, dt'. \tag{9}$$

Integration yields an equation for the generalized coordinate in steady state vibration, i. e.,

$$q_{mn} = \frac{F}{\varrho h \omega_{mn}} \frac{4}{ab} \sin(k_m x') \sin(k_n y') \frac{\omega_{mn}}{\omega_{mn}^2 - \omega^2} e^{-\mathrm{i}\omega t}. \tag{10}$$

Replacing q_{mn} in Eq. (7) by Eq. (10) and summing yields

$$w = \frac{F}{\varrho h} \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(k_m x) \sin(k_n y) \sin(k_m x') \sin(k_n y')}{\omega_{mn}^2 - \omega^2} e^{-\mathrm{i}\omega t} \,. \tag{11}$$

Thus, the time independent modal structural mobility is given by

$$Y_{mn} = \frac{v_{mn}}{F} = -\mathrm{i} \frac{4\omega}{\varrho hab} \frac{\sin(k_m x) \sin(k_n y)}{\omega_{mn}^2 - \omega^2} \sin(k_m x') \sin(k_n y') \,. \tag{12}$$

3 MATLAB-Example

In this example, the vibration velocity in z-direction is calculated using the equations from the former section. It is assumed that two forces with $F_\mathrm{d} = 1N$ are phase shifted to each other and exciting the thin plate, see Figure 1. In this Matlab-example, the plate's size is linked to acoustic wavelength.

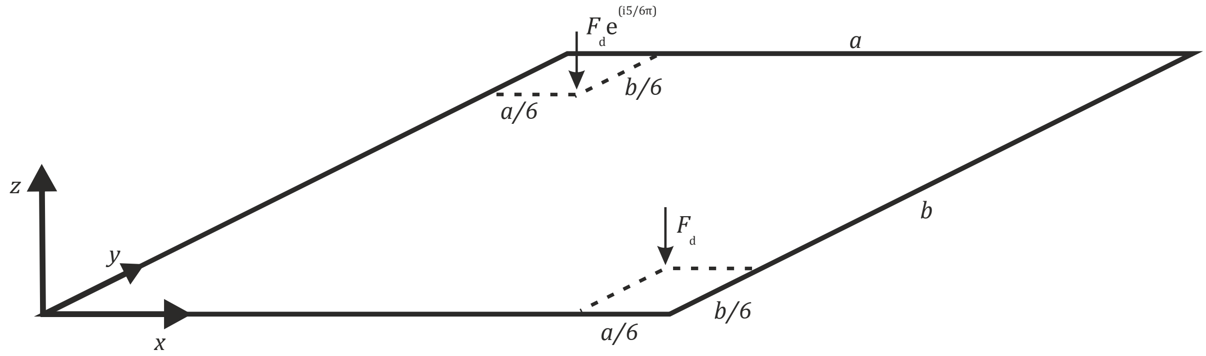

Figure 1: Schematic of the MATLAB-example

```
clear; clc; close all;
c0 = 343.2; % speed of sound [m s^-1]
freq = 1000; % frequency [s^-1]
w = 2*pi*freq; % angular frequency [rad s^-1]
k = w/c0; % angular wave number [rad m^-1]
a = 3*2*pi/k; % length of the plate [m]
b = 1.5*2*pi/k; % width of the plate [m]
h = 2*pi/k/25; % thickness of the plate [m]
fd = -1*[1 exp(1j*5/6*pi)]; % disturbance forces [N]
xd = [a/6 5*a/6]; yd = [5*b/6 b/6]; % coordinates of disturbance forces [m]
E = 210e9; % Young's modulus [N m^-2]
nu = 0.29; % Poisson's ratio
rhoS = 7850; % density of steel [kg m^-3]
D = E*h^3/12/(1-nu^2); % bending stiffness
nx = 50; ny = round(b/a*nx); % spatial resolution
x = linspace(a/2/nx,a-a/2/nx,nx); % spatial resolution
y = linspace(b/2/ny,b-b/2/ny,ny); % spatial resolution
[XX,YY] = meshgrid(x,y); X = XX(:); Y = YY(:); % spatial resolution
modeSpec = 1:500; % number of considered structural normal modes
%% sort angular eigenfrequencies
iVec = 0; % initial value
for m = 1:100 % mode indices
    for n = 1:100 % mode indices
        iVec = iVec + 1; % index number
        wmn(iVec) = sqrt(D/rhoS/h)*(m^2*pi^2/a^2+n^2*pi^2/b^2); % angular
eigenfrequencies [rad/s]
        mt(iVec) = m; % related mode indices
        nt(iVec) = n; % related mode indices
    end
end
[wmnSort,I] = sort(wmn); % sort
mSort = mt(I); % sorted mode indices
nSort = nt(I); % sorted mode indices
%% Normal vibration velocity
sdt = 0; % initial values
for ii = modeSpec % considered model spectrum
```

```
Wmn = sin(mSort(ii)*pi*x/a).*sin(nSort(ii)*pi*y/b).'; % modal displacement of
field points
wd = sin(mSort(ii)*pi*xd/a).*sin(nSort(ii)*pi*yd/b); % modal displacement of
excitation points
sdt = sdt+Wmn(:).*wd/(wmnSort(ii)^2-w^2); % temporary transfer value
end
vz = -1j*w*4/a/b/rhoS/h*fd*sdt.'; % velocity / time derivative of Eq. (11)
figure1 = figure; % plot
contourf(XX,YY,reshape(real(vz),ny,nx)); hold on % plot
xlim([0-h a+h]); ylim([0-h b+h]); % plot
title('vibration velocity (real) in [m/s]'); % plot
scatter(xd,yd,500,'x','MarkerEdgeColor',[1 1 1],'LineWidth',3); axis equal; % plot
% plot
xlabel('x'); ylabel('y'); colorbar; grid minor; % plot
xlabel('x in [m]'); ylabel('y in [m]'); % plot
set(gca,'fontname','garamond','FontSize',28); % plot
```

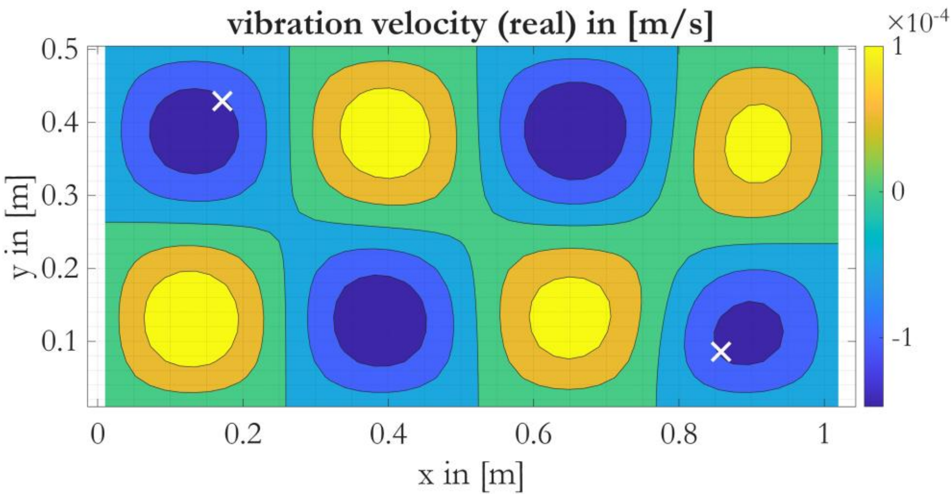

Figure 2: Vibration velocity from Eq. (11); white crosses mark force positions

4 Literature

[1] S. Ungnad, "DFG grant application: Near-field Acoustical Holography - a new sensor concept for methods of active noise reduction", 2025, doi: 10.24405/21742.

[2] S. Timoshenko, *Vibrations problems in engineering*. Constable & Co., 1928.